The volume of a truncated pyramid in ancient Egyptian papyri

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The Moscow Mathematical Papyrus, which was acquired by the Moscow Museum of Fine Arts from the Egyptologist Golenischeff in 1912, contains 25 problems of varying interest and importance. Problem No. 14 shows quite clearly that the Egyptian scribes were familiar with the formula.

$$V = \frac{h}{3}\,(a^2 + ab + b^2)$$

for the volume of a truncated pyramid (or frustrum), where h is the height and a and b are the edges of the square base and the square top, respectively.

While it has been generally accepted that the Egyptians were well acquainted with the formula for the volume of the complete square pyramid,

$$V = \frac{h}{3}\,a^2,$$

it has not been easy to establish how they were able to deduce the formula for the truncated pyramid, with the mathematics at their disposal, in its most elegant and far from obvious form, which, in the words of Gunn and Peet, "has not been improved on in 4000 years."

It would have been a simple enough operation to determine that a square pyramid made hollow had a capacity exactly one-third that of a cubical box of the same height, by merely pouring sand or water into them. By making them solid, using Nile mud or clay, the same result could have been achieved by weighing. Not so simple is the method of dissection—that of cutting up, say, a cube—to obtain three equal pyramids with the same square base and height. The only convincing dissection method known to me is to make six congruent "Juel" pyramids which fit together to form a cube. These would have a vertical height exactly half of the edge of the resulting cube, for a "Juel" pyramid has its sides sloping at 45°. The dissectionist Harry Lindgren, of Canberra, Australia, has communicated to me a method by which three models of a truncated "Juel" pyramid can be dissected into three slabs, $a \times a$, $b \times b$, $a \times b$, each of thickness h, which would establish the formula.

Earlier attempts to establish the formula as an Egyptian might have done it have been made by Gunn and Peet (with R. Engelbach), *Jr. Egypt. Arch.*, 1927; by Kurt Vogel, *Jr. Egypt. Arch.*, 1930; by P. Luckey, *Z. für Math. und Naturwiss.*, LXI, 1930; by W. R. Thomas, *Jr. Egypt. Arch.*, 1931; and more recently by Van der Waerden, *Science Awakening*, 1954.

In the specific case which we are considering, No. 14 of the Moscow Papyrus, the scribe sets the problem of finding the volume of a truncated square pyramid of height 6, base edge 4, and top edge 2 (see Fig. 1). The solution is correctly given as,

$$V=\frac{6}{3}[4^2+4\cdot 2+2^2]$$
$$=2[16+8+4]$$
$$=2\cdot 28$$
$$=56.$$

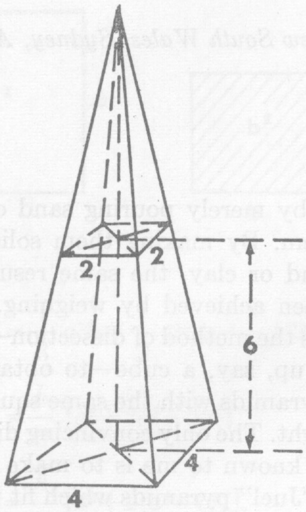

Figure 1

Assuming a knowledge of the formula for the volume of a pyramid, as one-third of the area of the base times the height, the question is, how may the scribe have derived the formula,

$$V=\frac{h}{3}[a^2+ab+b^2],$$

keeping in mind that a knowledge of the elementary algebra involved in the identities

$$a^3-b^3=(a-b)(a^2+ab+b^2),$$
$$a^2-b^2=(a-b)(a+b),$$

is nowhere attested in the extant Egyptian papyri; therefore, they may not be utilized.

The volume of the frustrum is the difference between the volumes of the original pyramid and the smaller pyramid cut off from the top. Then (see Fig. 2)

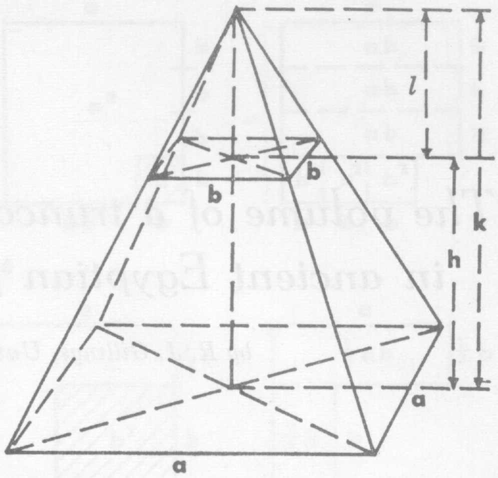

Figure 2

$$V=\frac{1}{3}a^2k-\frac{1}{3}b^2l$$
$$=\frac{1}{3}a^2(h+l)-\frac{1}{3}b^2l$$
$$=\frac{1}{3}a^2h+\frac{1}{3}a^2l-\frac{1}{3}b^2l. \qquad (1)$$

We take first the special case which the scribe gives in No. 14 of the Moscow Papyrus, where $a=2b$ and hence, by simple geometry, $l=h$. We then have, from (1),

$$V=\frac{1}{3}a^2h+\frac{1}{3}a^2h-\frac{1}{3}b^2h$$
$$=\frac{h}{3}[a^2+a^2-b^2]. \qquad (2)$$

We evaluate the last two terms in the brackets, a^2-b^2, which are the square bases of the original pyramid and the smaller pyramid cut from the top, by cutting the smaller square from the larger and examining the area left (see Fig. 3). Then the area

$$a^2-b^2=ab+b^2$$

so that, from (2),

$$V=\frac{h}{3}[a^2+ab+b^2].$$

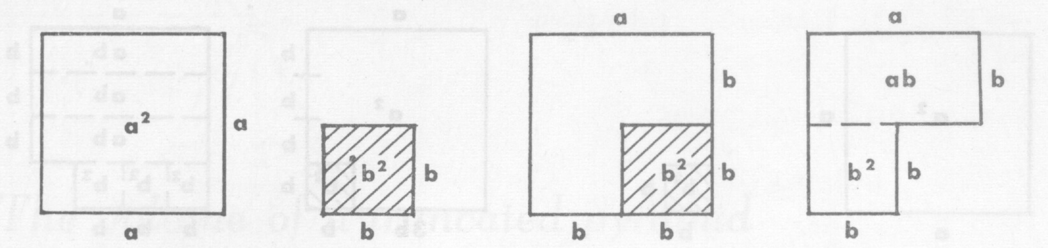

Figure 3

Now consider the case in which $a=3b$ and, again by simple geometry, $l=\frac{1}{2}h$. We then have, from (1),

$$V=\frac{1}{3}a^2h+\frac{1}{3}a^2\frac{1}{2}h-\frac{1}{3}b^2\frac{1}{2}h$$

$$=\frac{h}{3}\left[a^2+\frac{1}{2}(a^2-b^2)\right]. \qquad (3)$$

We evaluate the last two terms in the brackets, which equal half the difference of the top and bottom of the truncated pyramid, by cutting the smaller square from the larger, and again examining the remaining area (see Fig. 4). Then the area

$$a^2-b^2=2ab+2b^2,$$

so that

$$\frac{1}{2}(a^2-b^2)=ab+b^2,$$

and, from (3),

$$V=\frac{h}{3}[a^2+ab+b^2]$$

as before.

Consider further the case in which $a=4b$ and hence, as before, $l=\frac{1}{3}h$.

We have, from (1),

$$V=\frac{1}{3}a^2h+\frac{1}{3}a^2\frac{1}{3}h-\frac{1}{3}b^2\frac{1}{3}h$$

$$=\frac{h}{3}\left[a^2+\frac{1}{3}(a^2-b^2)\right]. \qquad (4)$$

We again evaluate the last two terms in the brackets, which equal one-third of the difference of the top and bottom of the truncated pyramid. We do this again by cutting the smaller square from the larger and examining the remaining area by elementary geometry (see Fig. 5). Then the area

$$a^2-b^2=3ab+3b^2,$$

so that

$$\frac{1}{3}(a^2-b^2)=ab+b^2,$$

and, from (4),

$$V=\frac{h}{3}[a^2+ab+b^2].$$

Clearly, this process can be continued for all cases where $a=nb$, where n has the integral values 2, 3, 4, 5, $\cdots$. Let us therefore consider a fractional value for n and assume $a=\frac{3}{2}b$ and hence, as before, $l=2h$. We have, from (1),

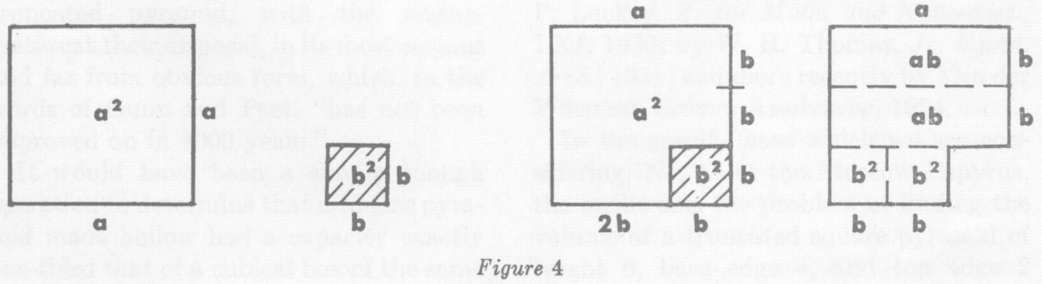

Figure 4

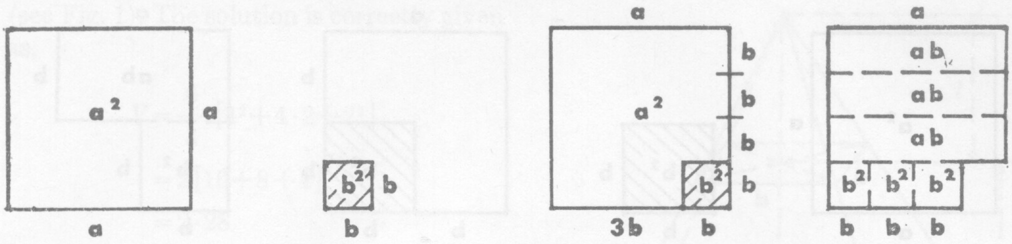

Figure 5

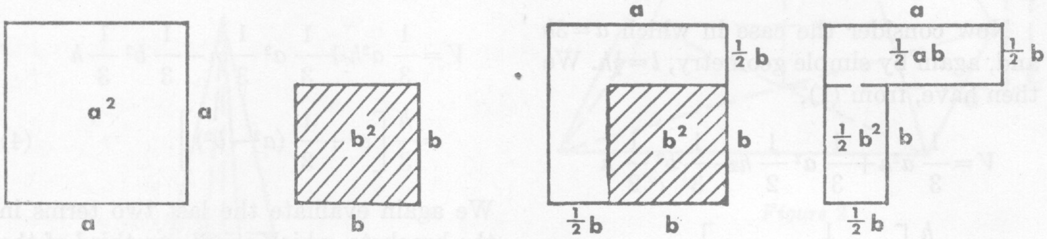

Figure 6

$$V=\frac{1}{3}a^2h+\frac{1}{3}a^22h-\frac{1}{3}b^22h$$

$$=\frac{h}{3}\,[a^2+2a^2-2b^2]$$

$$=\frac{h}{3}\,[a^2+2(a^2-b^2)]. \qquad (5)$$

We evaluate the last two terms in the brackets, $2(a^2-b^2)$, which in this case is twice the difference of the areas of the top and bottom of the truncated pyramid. By cutting the smaller square from the larger, we evaluate this by simple geometry (see Fig. 6). Then the area

$$a^2-b^2=\frac{1}{2}\,ab+\frac{1}{2}\,b^2$$

so that

$$2(a^2-b^2)=ab+b^2,$$

and, from (5),

$$V=\frac{h}{3}\,[a^2+ab+b^2].$$

The Egyptian scribe is now entitled to conclude inductively that, since the formula holds for the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . , and also for his familiar $\frac{2}{3}$ fraction, it therefore holds in all possible cases.

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