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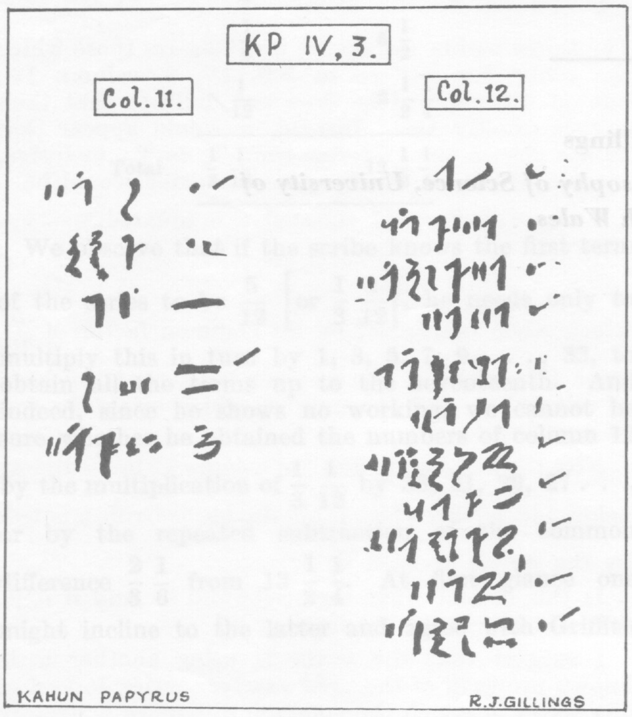
The Kahun Papyrus (KP), which was found by W. M. F. Petrie at Kahun, Egypt, in 1889, contains six mathematical fragments of interest to the historian of Egyptian Mathematics. The date of its origin is about the same as that of the Rhind Mathematical Papyrus (RMP) and the Egyptian Mathematical Leather Roll (EMLR), namely, *circa* 1800 B.C. In 1897 F. L. Griffith published *The Petrie Hieratic Papyri from Kahun and Gurob*, in which he gave facsimiles, with translations and discussions. Partly because some of the hieratic characters have lost their form, and partly because of the scribe's extreme brevity of writing, it has not been possible to interpret fully three of these problems, and of the intriguing fragment listed as Kahun, IV, 3, Griffith writes as follows.



'The number 110 having apparently been divided by 8, gives $13\frac{2}{3}\frac{1}{12}$. In the following lines $\frac{10}{12}$ is subtracted 9 times from $13\frac{2}{3}\frac{1}{12}$ and its successive remainders. Then $\frac{1}{3}\frac{1}{12}$ is multiplied by 9, giving $3\frac{2}{3}\frac{1}{12}$. I must confess I do not see the connexion between these operations, but probably they are all parts of one problem.'

In modern notation, this is what the scribe has written.

Line	Column 12	Column 11	
1	110	\1	$\frac{5}{12}$
2	$13\frac{3}{4}$	2	$\frac{5}{6}$
3	$12\frac{11}{12}$	4	$1\frac{2}{3}$
4	$12\frac{1}{12}$	\8	$3\frac{1}{3}$
5	$11\frac{1}{4}$	Total	$3\frac{3}{4}$
6	$10\frac{5}{12}$		
7	$9\frac{7}{12}$		
8	$8\frac{9}{12}$		
9	$7\frac{11}{12}$		
10	$7\frac{1}{12}$		
11	$6\frac{1}{4}$		

No further explanation is given, and since the preceding problem deals with the volume of a cylindrical granary, whose dimensions bear no relation whatever to the numbers of columns 11 and 12, we must regard KP, IV, 3, as a complete entity in itself.

We therefore address ourselves to the question, what mathematical problem, if any, was the scribe attempting to solve, and if we can determine this, what can we deduce regarding his methods? As far as I am aware, no historian of mathematics has hitherto resolved this enigma, and T. G. H. James, of the Department of Egyptian Antiquities (British Museum), Editor of the *Journal of Egyptian Archaeology*, tells me in a recent letter that he knows of no discussions on it.

At a first glance the problem seems to refer to some kind of an arithmetical progression (AP), and it is attested (RMP, problems 40, 64) that the Egyptians were familiar with many of the properties of AP's, and perhaps with their summation. The natural number series would of course provide the simplest examples of such progressions, as

1 2 3 4 5 6 . . ., *natural numbers*,
1 3 5 7 9 11 . . ., *odd numbers*,
2 4 6 8 10 12 . . ., *even numbers*,

and whatever properties the scribes may have established for these series, by a simple multiplication or division by a chosen common factor, they could produce another AP which would leave these properties unaltered.

It appears to me that the scribe of the KP has established the following properties for the '*odd number*' series,

1 3 5 7 9 11 . . .

just as the Greeks did more than a millennium later.

* * * *

(i) Sum of 2 terms $=1+3 = 4=2$ squared.
Sum of 3 terms $=1+3+5 = 9=3$ squared.
Sum of 4 terms $=1+3+5+7=16=4$ squared,

and so on indefinitely, and thus inductively he developed the rule that the sum of *n* terms is *n* squared.

(ii) The simple observation that

the 5th term, $T_5=(2\times5-1)=9$,
the 6th term, $T_6=(2\times6-1)=11$,
the 7th term, $T_7=(2\times7-1)=13$,

and so on, would lead him in a similar way to the general statement that the *n*th term is $(2\times n-1)$.

(iii) I am also convinced that the scribe knew that the '*odd number*' series was one in which the common difference was double of the first term, a unique property which he used on occasions.

From these three established properties it would be easy for the scribe to devise a means of finding the sum of any consecutive sequence of such series by a suitable subtraction. Thus suppose he required the sum of the 10 terms from the 8th to the 17th terms, inclusive. He can do this by finding from his rule (i) the sum to 17 terms and subtracting from it the sum to 7 terms, so that the answer he is seeking is

$S_{17}-S_7=17$ squared -7 squared,
$=289-49$,
$=240$.

I suggest that in KP, IV, 3, the scribe has been at pains to frame a problem, based on these properties and using these techniques, to bring them to the attention of his students or readers. First he must disguise or alter the basic '*odd numbers*', and he must choose a suitable multiplying factor for the series, which will not, however, unduly complicate the work with unmanageable fractions. And so he cleverly chooses the common factor of $\frac{5}{12}$, or as he writes it, using only unit fractions, $\frac{1}{3}\ \frac{1}{12}$. Then the sum of the 10 consecutive terms of his new series from T_8 to T_{17} inclusive, he knows at once is $\frac{5}{12}$ of 240, which is 100, and his AP now is as below, written, however, for convenience in modern notation instead of in Egyptian Unit Fractions.

T_1	T_2	T_3	T_4	T_5	T_6	T_7
$\frac{5}{12}$	$1\frac{1}{4}$	$2\frac{1}{12}$	$2\frac{11}{12}$	$3\frac{3}{4}$	$4\frac{7}{12}$	$5\frac{5}{12}$

[T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}]	T_{18} . . .
[$6\frac{1}{4}$	$7\frac{1}{12}$	$7\frac{11}{12}$	$8\frac{3}{4}$	$9\frac{7}{12}$	$10\frac{5}{12}$	$11\frac{1}{4}$	$12\frac{1}{12}$	$12\frac{11}{12}$	$13\frac{3}{4}$]	$14\frac{7}{12}$ . . .

The terms within the square brackets are those which the scribe has written in KP, IV, 3, column 12; but on the papyrus he does not indicate that their sum is exactly 100. Instead, it would appear that he has chosen the 12 terms from T_6 to T_{17} inclusive, the sum of which on the same reasoning would be

$S_{17}-S_5=17$ squared $-$ 5 squared,
$=289-25$,
$=264$,

which when multiplied by his common factor of $\frac{5}{12}$, gives 110, which is the number written at the head of column 12, and the number which originally was so puzzling to us.

Now the number on line 2 of column 12 immediately below 110, is $13\frac{3}{4}$, which is obtainable as Griffith noted, by the simple division by 8.

Is this in fact what the scribe did? And if it is, why did he choose the number 8?

I suggest that the scribe is using another little known property of the '*odd number*' series, indeed of any series in which the common difference is exactly double of the first term, which is that *the sum of the 12 terms from the sixth to the seventeenth, if divided by the seventeenth term, is always 8.*

Consequently, the same sum divided by the number 8 will always give the seventeenth term, and this is the division, the answer to which the scribe has written in line 2. If we are correct so far in unravelling the scribe's procedure, we now have to find out how he would explain the continual subtraction of $\frac{5}{6}$ from

T_{17} to his students; in other words, how to find the common difference of the series?

In the original '*odd number*' series the scribe knows from his rule (ii) that $T_{17}=(2\times17-1)=33$, and since he now knows T_{17} of his new series is $13\frac{3}{4}$, a simple division will give the common multiplying factor by which it is obtained, and this is $\frac{5}{12}$.

This number is also the value of the first term, since $1\times\frac{5}{12}=\frac{5}{12}$, and by his rule (iii) the common difference is double this, namely $\frac{5}{6}$, which is the number he will continually subtract from T_{17}, which he knows is $13\frac{3}{4}$.

Working for $13\frac{3}{4}\div33$ not shown by the scribe

	1	33
	$\frac{2}{3}$	22
	$\backslash\frac{1}{3}$	11
	$\frac{1}{6}$	$5\ \frac{1}{2}$
	$\backslash\frac{1}{12}$	$2\ \frac{1}{2}\ \frac{1}{4}$
Total	$\frac{1}{3}\ \frac{1}{12}$	$13\ \frac{1}{2}\ \frac{1}{4}$

We observe that if the scribe knows the first term of the series to be $\frac{5}{12}$ $\left[\text{or } \frac{1}{3}\ \frac{1}{12}\right]$, he needs only to multiply this in turn by 1, 3, 5, 7, 9, . . ., 33, to obtain all the terms up to the seventeenth. And indeed, since he shows no working, we cannot be sure whether he obtained the numbers of column 12 by the multiplication of $\frac{1}{3}\ \frac{1}{12}$ by 33, 31, 29, 27 . . ., or by the repeated subtraction of the common difference $\frac{2}{3}\ \frac{1}{6}$ from $13\ \frac{1}{2}\ \frac{1}{4}$. At first glance one might incline to the latter and agree with Griffith when he says, 'In the following lines, $\frac{10}{12}$ is subtracted 9 times from $13\ \frac{2}{3}\ \frac{1}{12}$', but then when we look at column 11 we see that the scribe has actually performed a multiplication of the first term $\frac{5}{12}$ $\left[\text{or } \frac{1}{3}\ \frac{1}{12}\right]$ by 9, which in fact gives him the fifth term, since from his rule (ii) it is $(2\times5-1)\times\frac{5}{12}$ namely $3\frac{3}{4}$. Indeed, from the multiplication shown in column 11, the scribe could, if he wished, have read off the sixth, seventh and eighth terms by adding the fractions on the R.H.S. of the multipliers 1 and 2 and 8 for 11 (T_6), the multipliers 1 and 4 and 8 for 13 (T_7), and the multipliers 1 and 2 and 4 and 8 for 15 (T_8).

I think that whatever problem was before the scribe in KP, IV, 3, there can be no doubt that some additions of terms were intended, for in the papyrus check marks are clearly discernible on the R.H.S. of lines 7 to 11, while others appear to be partly obliterated on the other lines. These check marks are standard in Egyptian arithmetic to indicate which numbers are to be included in any addition.

I conclude from the foregoing, therefore, that the scribe was solving, or was preparing for solution, the following problem.

> 'In a series of numbers in AP, whose common difference is double of the first term, the sum of the 12 consecutive terms from the 6th to the 17th is 110. What are the terms of this series?'

His method of solution was as follows, though briefer.

1. Divide 110 by 8, giving $13\frac{3}{4}$, which is T_{17}.
2. Divide $(2\times17-1)$ or 33 into T_{17} or $13\frac{3}{4}$, giving $\frac{5}{12}$, the first term.
3. Multiply the first term, $\frac{5}{12}$, by 2, giving $\frac{5}{6}$, the common difference.
4. Either (*a*) Continuously subtract $\frac{5}{6}$ from $13\frac{3}{4}$,
   or (*b*) Continuously multiply $\frac{5}{12}$ by 17, 15, 13, . . . 3.

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