To Prof. B.R.D'A. Gillings, reprint from Historia Mathematica, Vol. 6 N°4. Nov 1979, with my compliments. RJ Gillings Jan. 1980

CORRESPONDENCE

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THE RECTO OF THE RHIND MATHEMATICAL PAPYRUS AND THE EGYPTIAN MATHEMATICAL LEATHER ROLL

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Since its publication there have been up to 50 reviews of *Mathematics in the Time of the Pharaohs,* [Gillings, 1972], in various journals, in 6 different languages, in 14 different countries. More than 50 famous and well-known mathematicians have been quoted in these reviews. Some have written me useful letters regarding the book, but there have been few references to Chapters 6 and 7, which deal with the Recto of the *Rhind Mathematical Papyrus* (RMP), in which the scribe Ahmose used the first six feet of this 18-foot-long papyrus to make a complete table of 2 divided by each of the 50 odd numbers from 3 to 101. Each division is expressed as the sum of 2, 3, or 4 unit fractions with as few terms as possible, no denominator is as big as 1000, and *even* numbers are preferred to *odd* numbers. These equalities of the Recto, which are constantly utilised in the 87 problems of the remaining 18 feet of the papyrus, have interested historians since it was first translated in 1877.

Ahmose required so much space for his reference table because he took pains to prove that each of his equalities was correct; in the case of some of the larger prime numbers, the proofs required considerable space. He made no errors. His sole oversight, if it can be so-called, was his answer to $2 \div 95$, as

$$\frac{1}{60} + \frac{1}{380} + \frac{1}{570}$$

instead of

$$\frac{1}{60} + \frac{1}{228},$$

which is so much simpler. In recent years historians have written frequently on the preparation of his Recto table, for Ahmose did not show *how* he obtained his 50 answers, although he took up space to *prove* that they were correct. Mathematicians have shown increasing interest in determining the *method* which may have been used to prepare the table. We can not be certain whether Ahmose's method has been fully determined; even using modern mathematical techniques, algebraic formulas, and data from electronic computers, our conclusions are still not unanimous.

However, Ahmose has left one clue in the Recto, and that is in his division of 2 by 35. There he wrote a 6, *IN COLOUR RED*, near the 35, and next to it an ordinary black 7 and a 5, while the rest of his proof is done just as the other 49 divisions are. In my view, the red 6 means, that $6 \div 6$ is the multiplier for the fraction $2 \div 35$, and Ahmose's method, however he may have recorded it in his rough notes, must have amounted to something like the following:

$$\frac{2}{35} = \frac{2}{35} \times \frac{6}{6}$$

$$= \frac{12}{7 \cdot 5 \cdot 6}$$

$$= \frac{7 + 5}{7 \cdot 5 \cdot 6}$$

$$= \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 6},$$

and therefore, $$\frac{2}{35} = \frac{1}{30} + \frac{1}{42}.$$

But all that Ahmose wrote down in his Recto table was that $\overline{30}$ of 35 is 1 $\overline{6}$, and $\overline{42}$ of 35 is $\overline{\overline{3}}$ $\overline{6}$, showing that these operations were true and leaving the reader to verify that the sum of 1 $\overline{6}$ and $\overline{\overline{3}}$ $\overline{6}$ is 2, which is a very simple operation. Now we turn to what I have referred to as Ahmose's "oversight" in his division of 2 by 95. Using the method described above, with multiplier $12 \div 12$, yields:

$$\frac{2}{95} = \frac{2}{95} \times \frac{12}{12}$$

$$= \frac{24}{19 \cdot 5 \cdot 12}$$

$$= \frac{19 + 3 + 2}{19 \cdot 5 \cdot 12}$$

$$= \frac{1}{5 \cdot 12} + \frac{1}{19 \cdot 5 \cdot 4} + \frac{1}{19 \cdot 5 \cdot 6}.$$

Therefore $$\frac{2}{95} = \frac{1}{60} + \frac{1}{380} + \frac{1}{570},$$

which is of course correct.

He faltered, in the third line, where for 24 he should have substituted 19 + 5. This would have resulted in

$$\frac{1}{5 \cdot 12} + \frac{1}{19 \cdot 12},$$

giving him the shorter and simpler answer of

$$\frac{1}{60} + \frac{1}{228}.$$

Indeed, this division shows us that there is another substitution he could have made, namely,

$$\frac{2}{95} = \frac{20 + 4}{19 \cdot 5 \cdot 12} = \frac{1}{57} + \frac{1}{285}.$$

However, he would not have accepted this, because the denominators of the fractions are odd numbers, which he generally avoided.

We turn now to the 17 divisions of 2 by the multiples of 3. For simplicity we will write the equalities

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6},$$

etc., as

2 divided by 3 is 2 6
2 divided by 9 is 6 18
2 divided by 15 is 10 30 , etc.

In this group of divisions Ahmose's multiplier was 2, so that his method would take the form of

$$\frac{2}{27} = \frac{4}{27 \cdot 2} = \frac{3+1}{27 \cdot 2} = \frac{1}{18} + \frac{1}{54}.$$

We are unable to say whether Ahmose did his divisions in this order, or whether he observed the arithmetical progressions with common differences of 6, 4, and 12 in each column.

We consider next the 10 divisions of 2 by the multiples of 5. In this group his multiplying factor was 3 instead of 2, producing a similar set of 10 terms of an arithmetical progression:

2 divided by 5 is 3 15
2 divided by 15 is 9 45
2 divided by 25 is 15 75, etc.

But Ahmose must have been disappointed that the terms, although in pairs and not so large, were odd numbers, which he wished to avoid whenever possible. However, the numbers 15, 45, and 75 are also multiples of 3 and had already been included in that group. He then tries the multiplier 6, as in the case of 2 ÷ 35 (already discussed). The same multiplier works in

$$\frac{2}{55} = \overline{30}\ \overline{330}\ ,$$

but for 2 ÷ 95 he used the multiplier 12 which avoided $\overline{57}\ \overline{285}$. However, as noted earlier, he faltered here and recorded $\overline{60}\ \overline{380}\ \overline{570}$, instead of $\overline{60}\ \overline{228}$. This oversight on Ahmose's part has been a great help to contemporary arithmeticians endeavouring to reconstruct his methods.

The divisions of 2 by odd multiples of 7 (up to 91) are carried out with a multiplying factor of 4. Once more the columns are in arithmetic progression:

2 divided by	7	is	4	28	
2 divided by	21	is	12	84	
2 divided by	35	is	20	140	, etc.
.	.		.	.	
.	.		.	.	
.	.		.	.	
2 divided by	91	is	52	364 .	

Some of these equalities have already been dealt with as multiples of 3 and 5 (2 divided by 21, 35 and 63). In his attempt to decrease the denominator 364, Ahmose went a step forward with the last division of the series. He succeeded in doing this by changing the multiplier from 4 to 10; thus,

$$\frac{2}{91} = \frac{20}{13 \cdot 7 \cdot 10} = \frac{13 + 7}{13 \cdot 7 \cdot 10} = \frac{1}{70} + \frac{1}{130}\ .$$

Finally we come to the divisions of the remaining 22 primes, and it is here that Ahmose's difficulties really began. The multipliers of 2, 3 and 4, which he had used earlier, were too small for the larger primes. Yet for his largest prime of all, he had only one possible multiplier, namely, 6, (already used in 2 ÷ 35). This yields

$$\frac{2}{101} \times \frac{6}{6} = \frac{12}{101 \cdot 3 \cdot 2} = \frac{6+3+2+1}{101 \cdot 3 \cdot 2} = \frac{1}{101} + \frac{1}{202} + \frac{1}{303} + \frac{1}{606}\ .$$

And now we note that the whole of the Recto table could be similarly written out, and that the arithmetical progressions would again appear, making it a simple matter to produce it by ordinary addition [see Gillings 1972, 69]:

2	divided by	3	is	3	6	9	18	
2	divided by	5	is	5	10	15	30	
.		.		.	.	.	.	
.		.		.	.	.	.	
.		.		.	.	.	.	
2	divided by	101	is	101	202	303	606	.

But Ahmose would have none of it, except of course the very last. Instead of 200 numbers, his finished table has only 129 numbers of which 105 are even numbers, and only 24 are odd. This was apparently much more to his liking, despite the great efforts to achieve it.

Searching for the best and appropriate multiplier for each of the 22 remaining primes must have been a laborious task, as we ourselves have today discovered, even using a computer. Here is one example:

$$\frac{2}{17} = \frac{2 \cdot 12}{17 \cdot 12} = \frac{24}{17 \cdot 4 \cdot 3} = \frac{17 + 4 + 3}{17 \cdot 4 \cdot 3} = \overline{12}\ \overline{51}\ \overline{68}\,.$$

I have checked all the multipliers which, it would appear from this analysis, Ahmose must have chosen for the 22 remaining primes of the Recto. The results are given in the following table:

The multiplier	No. of times used	For 2 divided by
The multiples of 3, 6	2	11 and 101
12	2	17 and 23
24	3	29 37 and 41
30	2	47 and 53
36	1	59
42	1	43
60	4	73 79 83 and 89
The multiples of 2, 8	1	13
56	1	97
The multiples of 5, 10	1	19
20	1	31
40	3	61 67 and 71

It would be interesting to the student of the Recto of the *Rhind Mathematical Papyrus* (RMP) to compare its earlier entries with the *Egyptian Mathematical Leather Roll* (EMLR). Both the EMLR and the RMP were purchased at Luxor in 1858 by the Scotsman A. H. Rhind. The EMLR remained unrolled until 1927 because of its brittle condition. It contains two identical tables, each of 26 additions of unit fractions similar to those in the Recto, and nothing else. There is no explanatory detail as in the Recto. Consider, for example, the seven equalities:

$$\overline{9}\ \ \overline{18} = \overline{6}$$
$$\overline{21}\ \ \overline{42} = \overline{14}$$
$$\overline{4}\ \ \overline{12} = \overline{3}$$
$$\overline{10}\ \ \overline{40} = \overline{8}$$
$$\overline{7}\ \ \overline{14}\ \ \overline{28} = \overline{4}$$
$$\overline{18}\ \ \overline{27}\ \ \overline{54} = \overline{9}$$
$$\overline{25}\ \ \overline{15}\ \ \overline{75}\ \ \overline{200} = \overline{8}.$$

It would be of great interest to determine how the scribe obtained the last four-termed and unique equality. Note that the $\overline{25}$ preceding the $\overline{15}$ is not an error in copying.

REFERENCES

Chace, A. B., Bull, L., Manning, H. P., and Archibald, R. C., [RMP] 1927. *The Rhind Mathematical Papyrus*, Vol. 1. Oberlin, Ohio: Mathematical Association of America.

Gillings, R. J. 1972. *Mathematics in the Time of the Pharaohs*. Cambridge, Mass.: MIT Press.

Glanville, S. R. K. 1927. The mathematical leather roll in the British Museum. *Journal of Egyptian Archaeology* 23.

Scott, A., and Hall, H. R. [EMLR] 1927. The Egyptian mathematical leather roll. Egyptian leather roll of the 17th century B.C. *British Museum Quarterly* 2.