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# TESTS OF DIVISIBILITY

R. J. Gillings

YESHIVA UNIVERSITY
Amsterdam Avenue and 186th Street, New York 33 N.Y.

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By R. J. Gillings

To test whether a given number is divisible by 2, it is only necessary to observe whether the end or units digit is even, and to test its divisibility by 5, it is only necessary to observe whether the end digit is 5 or 0. Thus the repeated division of any given number by 2 and 5 until the end digit is other than, 0, 2, 4, 6, 8, or 5, reduces the significant cases, to numbers ending in 1, 3, 7, or 9.

Therefore for the purpose of compactly stating tests of divisibility of all integers by any other integer, it is only necessary to confine ourselves to divisors ending in 1, 3, 7, or 9, and if for convenience we separate these divisors into the following four groups, it is clear that this array comprehends all significant cases.

| | | | | | | |
|---|---|---|---|---|---|---|
| *Group A* | 11 | 21 | 31 | 41 | 51 | 61 . . . . . . . . ∞ |
| *Group B* | 3 | 13 | 23 | 33 | 43 | 53 . . . . . . . . ∞ |
| *Group C* | 7 | 17 | 27 | 37 | 47 | 57 . . . . . . . . ∞ |
| *Group D* | 9 | 19 | 29 | 39 | 49 | 59 . . . . . . . . ∞ |

Then the tests are:

*Group A* For any divisor of the sequence, 11 21 31 41 51 61 . . . ,
multiply the end digit by, 1 2 3 4 5 6 . . . ,
and subtract the product from the remaining part of the number.

*Group B* For any divisor of the sequence, 3 13 23 33 43 53 . . . ,
multiply the end digit by, 1 4 7 10 13 16 . . . ,
and add the product to the remaining part of the number.

*Group C* For any divisor of the sequence, 7 17 27 37 47 57 . . . ,
multiply the end digit by, 2 5 8 11 14 17 . . . ,
and subtract the product from the remaining part of the number.

*Group D* For any divisor of the sequence, 9 19 29 39 49 59 . . . ,
multiply the end digit by, 1 2 3 4 5 6 . . . ,
and add this product to the remaining part of the number.

Then, the division will in all cases be exact, if on reiteration of the procedure, the final result is either zero, or a multiple of the divisor being tested.

* * *

Let the number $N$ be written $10n_1 + n_0$, where $n_0$ is the end or units digit, and $n_1$ is the remaining part of the number.

A. Then if, $10n_1 + n_0 \equiv 0 \pmod{(10x + 1)}$ $\quad x = 1, 2, 3, \ldots,$
so also will, $10n_1 + n_0 - (10x + 1)n_0 \equiv 0$
i. e., $10n_1 - 10xn_0 \equiv 0$
or, $10(n_1 - xn_0) \equiv 0$
so that, $n_1 - xn_0 \equiv 0 \pmod{(10x + 1)}$ $\quad x = 1, 2, 3, \ldots,$

Therefore $x$ times the end digit subtracted from the remaining part of the number also has a factor $(10x + 1)$. Then for divisors, 11 21 31 41 51 . . . . ∞ the multipliers are, seriatim, 1 2 3 4 5 . . . . ∞.

*Example.*

To test the divisibility of 68403 by 151, the multiplier is 15, and we proceed:

| | |
|---|---|
| | 68403 |
| 3 × 15 subtracted from 6840 | 45 |
| | 6795 |
| 5 × 15 subtracted from 679 | 75 |
| | 604 |
| 4 × 15 subtracted from 60 | 60 |
| | 0 |

Then the prime 151 is a factor of 68403.

B. Again if $10n_1 + n_0 \equiv 0 \pmod{10x - 7}$ $\quad x = 1, 2, 3, \ldots,$

so also will $10n_1 + n_0 + 3(10x - 7)n_0 = 0$

i. e., $10n_1 + n_0 + 30xn_0 - 21n_0 = 0$

or $10(n_1 + (3x - 2)n_0) = 0$

so that $n_1 + (3x - 2)n_0 = 0 \quad (\text{mod } (10x - 7) \quad x = 1, 2, 3, \ldots,$

Therefore $(3x - 2)$ times the end digit added to the remaining part of the number also has a factor $(10x - 7)$. Then for divisors, 3 13 23 33 43 . . . ∞ the multipliers are seriatim, 1 4 7 10 13 . . . ∞.

To test the divisibility of 12167 by 23, the multiplier is 7, and we proceed:

| | |
|---|---|
| | 12167 |
| 7 × 7 added to 1216 | 49 |
| | 1265 |
| 5 × 7 added to 126 | 35 |
| | 161 |
| 1 × 7 added to 16 | 7 |
| | 23 |

Then 23 is a factor of 12167.

C. Again if $10n_1 + n_0 \equiv 0 \pmod{10x - 3}$ $\quad x = 1, 2, 3, \ldots,$

so also will $10n_1 + n_0 - 3(10x - 3)n_0 = 0$

i. e., $10n_1 + n_0 - 30xn_0 + 9n_0 = 0$

or $10(n_1 - (3x - 1)n_0 = 0$

so that $n_1 - (3x - 1)n_0 = 0 \quad (\text{mod } (10x - 3)) \quad x = 1, 2, 3, \ldots,$

Therefore $(3x - 1)$ times the end digit subtracted from the remaining part of the number also has a factor $(10x - 3)$. Then for the divisors, 7 17 27 37 47 . . . . ∞ the multipliers are seriatim, 2 5 8 11 14 . . . . ∞.

To test the divisibility of 50653 by 37, the multiplier is 11, and we proceed:

| | |
|---|---|
| | 50653 |
| 3 × 11 subtracted from 5065 | 33 |
| | 5032 |
| 2 × 11 subtracted from 503 | 22 |
| | 481 |
| 1 × 11 subtracted from 48 | 11 |
| Then 37 is a factor of 50653 | 37 |

D. Again if $10n_1 + n_0 \equiv 0 \quad (\text{mod } (10x - 1)) \quad x = 1, 2, 3, \ldots,$
so also will $10n_1 + n_0 + (10x - 1)n_0 \equiv 0$
i. e., $10n_1 + 10xn_0 \equiv 0$
or $10(n_1 + xn_0) \equiv 0$
so that $n_1 + xn_0 \equiv 0 \quad (\text{mod } (10x - 1)) \quad x = 1, 2, 3, \ldots,$

Therefore $x$ times the end digit added to the remaining part of the number also has a factor $(10x - 1)$. Then for divisors, 9 19 29 39 49 . . . . . ∞ the multipliers are, seriatim, 1 2 3 4 5 . . . . . . . ∞.

*Example.*

To test the divisibility of 24389 by 29, the multiplier is 3, and we proceed:

| | |
|---|---|
| | 24389 |
| 9 × 3 added to 2438 | 27 |
| | 2465 |
| 5 × 3 added to 246 | 15 |
| | 261 |
| 1 × 3 added to 26 | 3 |
| | 29 |

Then 29 is a factor of 24389

Tests of divisibility usually suffer from their impracticability with the larger numbers. If we try the interesting number, 1111111 consisting of seven ones, we observe that since the end digit is 1, then the end digits of its factors, if they exist, are both 9's or 3 and 7, two 1's being excluded for a simple reason. We therefore first try the primes ending in 9, other divisors ending in 9 being excluded also for a simple reason, and we succeed after 12 attempts with 15 minutes figuring.

For the divisor 239, the end digit multiplier is 24, and we have,

$$\begin{array}{r} 1111111 \\ 24 \\ \hline 111135 \\ 120 \\ \hline 11233 \\ 72 \\ \hline 1195 \\ 120 \\ \hline 239 \end{array}$$

By division we find the other factor to be the prime 4649.

Sydney University
Newtown, Sydney, Australia