

PIERRE DE FERMAT

R. J. GILLINGS

Pierre de Fermat died in 1665 at the age of 64, just three centuries ago. Now because we are to speak of a tercentenary event of high interest to mathematicians, and because we are to try to evaluate Fermat's place in the history of mathematics, perhaps we shall avoid committing the easy error of reciting a jumble of more or less meaningless dates, by noting instead where some of his contemporaries were placed, in relation to the significant year 1665, whose tercentenary we are celebrating.

Blaise Pascal, associated with Fermat as a pioneer in the study of Probability, predeceased him by only three years. Descartes, with whom he disputed over Analytical Geometry, had been dead for 15 years. Torricelli, of atmospheric pressure fame, died 18 years earlier, Galileo 23 years earlier, Kepler, the planetary astronomer, 35 years earlier, and Napier, the inventor of logarithms, 48 years earlier.

On the opposite side of this point of reference in the chronological frame, Halley, whose name was given to the famous comet, was nine years old, Leibniz, associated with Newton in originating the modern calculus ("the D-ism of Leibniz and the Dot-age of Newton!"), was 19 years old, Isaac Newton, afterwards Sir Isaac, the greatest mathematician of them all, was 22, Christopher Wren, the designer of St. Paul's, was 33, and Huygens was 36. Euler was not born until 42 years later, and that great master Carl Friederich Gauss did not see the light of day until more than a century later. Even without Gauss, what a century was the 17th for mathematicians!

During Fermat's lifetime, the Thirty-Years' War began and ended; Queen Elizabeth I (1603) and Shakespeare (1616) died; Charles I (1649) and Raleigh (1618) were beheaded; Louis XIV became King of France (1643); the Romanoff dynasty of Russia was founded (1613); the Stuarts became Kings of England; the Gunpowder Plot was foiled (1605); the Great Plague broke out in London (1665); and the Authorised Version of the Bible appeared (1611).

Pierre Fermat was one of the world's famous mathematicians! Now consider the present world population of nearly 2,000 million, and then ponder on how many people who might have been mathematicians must have lived on earth over the last 2,000 years, then think what this statement connotes. "One of the world's greatest" is a sweeping statement, yet no historian of mathematics would deny or even qualify it, because it is true.

And yet, what has Fermat left for the present day to read and marvel over, as (for example) the three greatest mathematicians who ever lived have done? We refer of course to Newton, Gauss and Archimedes, whose written works are justly famous. What has Fermat written at all comparable? Materially nothing! In this respect Fermat is unique. If we look in a catalogue or index of any reputable library, can we find any book he wrote, or mathematical text he published, or even his memoirs? We cannot. During his lifetime he published nothing, either as a bound volume or as a contribution to a learned journal, no articles, brochures or communications to the learned societies of France, England or Germany, not even to the famous French Academy. He did not conduct classes, he was never a professor, and he did not lecture either publicly or in colleges or universities or institutions.



Sketch of Fermat made from D. E. Smith's *Portraits of Eminent Mathematicians*, *Scripta Mathematica*, New York, and reproduced with the kind permission of The Editor, *Scripta Mathematica*, Yeshiva University.

In the light of all this, he was surely entitled to be referred to as The Great Amateur, "The Prince of Amateurs", as Eric Bell called him (*Men of Mathematics*, Dover, N.Y., 1937), and he goes on to describe him as "One of the foremost amateurs in the history of science, if not the very first". One cannot doubt Fermat's uniqueness as a mathematician.

One therefore opens Coolidge's more recent publication, *The Mathematics of Great Amateurs* (Oxford, 1949), expecting to find Fermat prominently featured therein. Among the 16 amateurs included in Coolidge's book we find names already famous in other fields, Leonardo da Vinci, Omar Khayyam, Albrecht Dürer, Buffon, Pascal and Diderot, but NO FERMAT! "Is this an accidental omission?" we think—"it can't be". Nor is it. Coolidge deliberately omits Fermat, "Because", he writes in his preface, "he was so truly great that he should count as a professional!" And Coolidge certainly knew his mathematicians.

How, then, can Fermat by common consent be so highly placed in history in a mathematician's order of merit list? Why is he so important? What did he do in the field of mathematics? What, in fact, do mathematicians do? Or, finally, what is mathematics? These questions must be answered.

Memorabilia Mathematica (1914), newly published under the title *On Mathematics and Mathematicians* (Moritz, Dover, N.Y., 1942), gives 35 definitions of Mathematics. Most of these are long and somewhat complex. A few are short and succinct, like

"Mathematics is the science which draws necessary conclusions." (Peirce.)

"Mathematics in its widest signification, is the development of all types of formal, necessary, deductive reasoning." (Whitehead.)

We do not, however, find the even shorter definition, phrased no doubt "en caprice", the authorship of which I am unaware, which states:

"Mathematics is what mathematicians do." (Anon.)

Had this intriguing definition been included, the reader might reasonably have pursued the thought with the further question:

"But what *do* mathematicians do?"

And this is a question which I am quite prepared to answer, and I answer

"They write mathematical textbooks!"

And if you accept this, then you must certainly agree that Fermat was again unique in his field. He wrote no textbooks!

Take one of the famous mathematical journals, *The Mathematical Gazette* (London), and choose at random any volume, as I did, Vol. XLIII, 1959. One finds that it contains 23 articles, 54 mathematical notes, and 174 reviews of recent mathematical textbooks, which by sheer bulk alone take up more than one-third of the 200 odd pages of Vol. XLIII. In addition, there are listed 141 titles as books received and awaiting attention, while 11 others get "brief mention". There were so many they would not fit in. Mathematicians certainly write mathematical textbooks!

And their titles are getting longer. Once they read *Modern Geometry*, *Higher Algebra* or *Elementary Trigonometry*. But now we are confronted by *Statistical Independence in Probability Analysis and Number Theory*, or *Integral Operators in the Theory of Partial Differential Equations* or *Vorlesungen über Differential und Integralrechnung Funktionen einer Variablen*, Vol. 1.

What would Fermat have thought of such a spate of mathematical outpourings? "Volume I indeed!" he would probably have said, in French, deprecatingly. He was a man of few words and fewer letters. He wrote no texts.

Nevertheless, every one of these numerous, expensive and expansive books, if it deals with Probability, Analytical Geometry, the Theory of Numbers, Maxima and Minima and the Calculus, Reflection and Refraction of Light, or Indeterminate Equations, owes something of its origins to Pierre de Fermat. He may not have written any texts himself, but he must surely be regarded as foremost among the pioneers of much of the matter of which they treat. The amateur we cannot forget!

Pierre de Fermat was born near Toulouse, France, in August, 1601, the son of a leather merchant, Dominique Fermat, and Clair de Long. He studied law in that city, became a lawyer, was made a Commissioner of Requests and married, at age 30, Louise de Long, a relative from his mother's family. In 1648 he became a King's Councillor for the provincial parliament of Toulouse, a position described by Kline as similar to that of a modern United States Attorney, while Hutton says that he had a high reputation as an enlightened judge. He had three sons and two daughters, who both became nuns. One of his sons was his executor and published his father's posthumous *Oeuvres*, including his extant letters to Mersenne, Carcavi, Roberval, Pascal, Descartes and many other mathematicians.

He was a student of languages and literature, devoting practically all his evenings to study, reading and writing verse, with no special preference for mathematics until Bachet's translation of Diophantus came into his hands.

It was his famous marginal note in this book which probably stirred his dormant mathematical genius to blossom, at the relatively late age of about 30. This note is now paradoxically known as *Fermat's Last Theorem*, still not completely proven. Of the Greek mathematician Diophantus's great work called *Arithmetic*, only six of its 13 books survive. They contain over 130 problems dealing with integers and their powers, satisfying given conditions, and the equations for solution require considerable skill in their manipulation. We know them now as *Indeterminate equations*, or as problems in *Diophantine Analysis*. In Bachet's 1621 translation of Diophantus's *Arithmetic*, problem 8 of Book II asks for the decomposition of a square into two smaller perfect squares, which means, find integral solutions of the equation

$$x^2 + y^2 = n^2$$

an arithmetical form of Pythagoras' Theorem.

In the margin of his copy, Fermat wrote: "However it is impossible to write a cube number as the sum of two other cubes, or indeed any power at all, as the sum of two like powers above 2."

In modern notation we may state *Fermat's Last Theorem* as

$$x^n + y^n = z^n,$$

is impossible in integers for any value of n above 2".

The ingenious methods devised by Diophantus for the problems he proposed, in which he used his own special signs, initial letters for operations and unknowns, and abbreviations for concepts in the solutions, has been called *syncopated algebra*. Diophantus (perhaps A.D. 75) flourished in the first century of our era. Only the Greek mathematicians flourish in the histories of the subject. Others just lived. If this was because of doubts as to their exact births and deaths, then we can allow Fermat to flourish also, for the date of his birth has been variously quoted as "late 16th century", 1601, 1608, "the commencement of the 17th century".

Fermat was a Frenchman, that is to say he came from the country which has produced more mathematicians of note than any other country in the world, and if, as Hogben has said (*Mathematics for the Million*, Allen and Unwin, London, 1936), "Mathematics is the mirror of civilization", then France should be one of the most civilized countries in the world. Perhaps it is.

Of the 32 chapters in Bell's *Men of Mathematics*, each devoted to an outstanding mathematician, there are 11 Frenchmen, 9 Germans, 5 Britons, 3 Greeks and a few others from Norway, Russia and Switzerland. Of the many lesser lights referred to in the body of the book, the great majority are French. In Turnbull's *The Great Mathematicians* (Methuen, London, 1951), a less ambitious volume, the same proportions obtain, although understandably there are more Englishmen. Eric Temple Bell is an American born in Scotland. France is the foremost nation, considered mathematically, and Fermat is still undoubtedly in the front rank of French mathematicians.

Kline (*Mathematics—A Cultural Approach*, Addison Wesley, Reading, Mass., 1962) says (p. 273), "Fermat's mathematical achievements entitle him to the honour of being considered one of the best mathematicians the world has had".

For obvious reasons it is not possible to record here Fermat's mathematical achievements, either chronologically or in order of importance. We will mention them briefly, "just as they come", from the records, from the letters he wrote, and from the many textbooks in which his name crops up from time to time, and from subject to subject.

THEORY OF NUMBERS

Every prime number of the form $(4n+1)$ is the sum of two squares in one way only.

$$\begin{aligned} 13 &= 4 \times 3 + 1 = 2^2 + 3^2, \\ 29 &= 4 \times 7 + 1 = 2^2 + 5^2, \\ 41 &= 4 \times 10 + 1 = 4^2 + 5^2, \text{ etc.} \end{aligned}$$

Fermat stated this theorem in a letter to Carcavi in 1659, saying how he could prove it, but without the actual proof. The first extant proof was given by Euler in 1749, after trying on and off for seven years.

If n and p are integers and p is a prime number, then $(n^p - n)$ is divisible by p .

This theorem was stated by Fermat without proof, and it appears to be a very useful theorem in other branches of mathematics. Bell says that it is within the understanding of any normal 14-year-old boy, but that only about one in a million would evolve a proof within a year. What a challenge that is!

$$\begin{aligned} 5^3 - 5 &= 120 = 3 \times 40 \\ 2^{11} - 2 &= 2046 = 11 \times 186 \\ 3^5 - 3 &= 240 = 5 \times 48 \end{aligned}$$

Leibniz proved this in 1683.

From Hutton's *Recreations in Mathematics* (Thos. Tegg, London, 1840), translated from Montucla's edition of *Ozanam*, we read (p. 69), "M. de Fermat was able to demonstrate the following general and curious properties of numbers, viz.:

"Every number is either triangular, or composed of 2 or 3 triangular numbers.

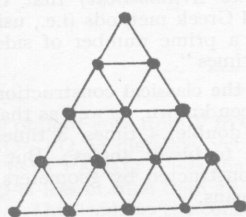
Every number is either square, or composed of 2, 3 or 4 square numbers.

Every number is either pentagonal, or composed of 2, 3, 4 or 5 pentagonal numbers.

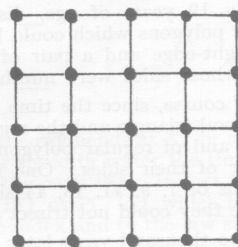
And so of the rest. A demonstration of these properties of numbers, if they be real, would be truly curious."

If Charles Hutton could have proved them it is certain he would have shown how in his *Recreations*.

Triangular numbers are 3, 6, 10, 15, 21 ..., being such that, for example, 15 counters can be arranged to form a triangle



15.



25.

and square numbers, e.g., 25, arranged to form a square.

Gauss proved the theorem for triangular numbers in 1796, and Cauchy proved it for square and pentagonal numbers in 1815.

Fermat gave no proof as usual for the simple statement that the equation

$$x^3 = y^2 + 2$$

has only one solution in integers.

Fermat's Method of Factorizing Large Numbers

Suppose it is known that the number 2627 is composite, and no factor tables are at hand. Fermat finds its factors as follows.

The square root of 2627 is between 51 and 52.

Now $52^2 = 2704$, and $2704 - 2627 = 77$, which is not a square.

Then $53^2 = 2809$, and $2809 - 2627 = 182$, which is not a square.

Then $54^2 = 2916$, and $2916 - 2627 = 289$, which is a square!

$$\begin{aligned} \text{Therefore we have } 54^2 - 2627 &= 17^2 \\ \text{or } 2627 &= 54^2 - 17^2 \\ &= (54 - 17)(54 + 17) \\ &= 37 \times 71, \end{aligned}$$

a neat application of the factors of the difference of two squares.

Another Fermat Theorem

The formula $(2^{2^n} + 1)$ for integral values of n , Fermat thought, but did not categorically state, gave all prime numbers. Thus

$$2^{2^0} + 1 = 2 + 1 = 3$$

$$2^{2^1} + 1 = 4 + 1 = 5$$

$$2^{2^2} + 1 = 16 + 1 = 17$$

$$2^{2^3} + 1 = 256 + 1 = 257$$

$$2^{2^4} + 1 = 65536 + 1 = 65537$$

All these numbers are primes, but the next number is far too large to be tested for primality by ordinary mortals. It is

$$4,294,967,297.$$

Years later it was shown to have a factor, 641, by Euler. This was in 1739, just a century after Fermat had made his conjecture that the number was prime. Interest in these "Fermat Primes" naturally lapsed, because they were not all primes, but 30 years later attention was dramatically re-directed to them when Gauss, scarcely 19 years of age, discovered (*Disquisitiones Arithmeticae*) that the only regular polygons which could be drawn by classical Greek methods (i.e., using only a straight-edge and a pair of compasses), having a prime number of sides, were those whose sides were numbered as "Fermat Primes".

Of course, since the time of Euclid (300 B.C.), the classical construction of the equilateral triangle and the regular pentagon had been known, as well as that of the square and of regular polygons, whose sides were double, 4 times, 8 times, ... the number of their sides. One had only to be able to bisect angles. But regular polygons of 7, 9, 11, 13, 14 sides were never so constructed by geometers, partly because they could not trisect angles by classical means.

Two thousand years later a German boy of 19 proved that regular polygons of 3, 5, 17, 257 and 65,537 sides could be so constructed. We do not know whether Gauss ever found out how to construct a regular 17-gon, or 257-gon, or 65,537-gon, but other geometers have since tried. For example, the 17-gon and its construction is very thoroughly treated by L. E. Dickson in *Monographs on Modern Mathematics*, Ed. Young (Longmans, Green, N.Y., 1924). It is tough going.

Fermat's Last Theorem

(We do not know why it is designated his *Last*), we have already spoken of that. And still a complete proof that the equation

$$x^n + y^n = z^n,$$

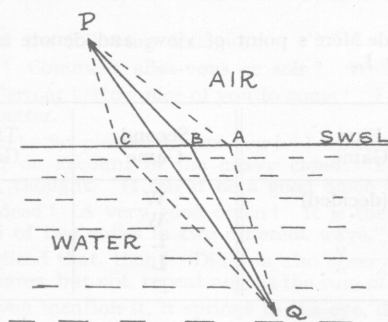
is impossible in integers for any value of n greater than 2 has not been found.

Fermat probably had a proof for $n=3$, a challenge which he repeatedly threw out to other mathematicians. He certainly had a proof for $n=4$. The first published proof for $n=3$ was given by Euler, $n=5$ by Dirichlet and Legendre independently about 1825, and $n=7$ by Lamé in 1839. From the basic work of Kummer, 1843, proofs for n up to 600 have been found and, according to Bell, for all primes up to $n=14,000$. More recent work by Vandiver, Lehmer, Nicol, Selfridge, Oblath and Inkeri verify that the theorem is true for all n up to 4,000 (1954) and perhaps by now with electronic computers this has been further extended.

In 1908 the German mathematician Wolfskehl, who had made some contributions to the study of Fermat's Last Theorem, bequeathed the sum of 100,000 Deutschmarks to the Göttingen Academy of Science to be awarded for the first complete proof of the theorem. The prize was never awarded. So many faulty alleged proofs were received, from people quite unequipped to appreciate the inherent difficulties, that it must surely rival the famous earlier problem, Squaring the Circle, as being the mathematical problem for which the greatest number of incorrect proofs have been written. In 1775 the French Academy refused to examine any further solutions of the quadrature, now known (since Lindemann in 1882 proved π was transcendental), to be impossible. But the Göttingen Academy of Science is presumably still open for business, although since two world wars the prize money must now be negligible.

LAWS OF REFRACTION

In the theory of the laws of reflection and refraction, Fermat found what he called the principle of least time.



If a ray of light goes from P to Q, from air to water, the ray bends at A, for although $PA + AQ > PQ$, in distance, yet because of the slower speed in water the time taken by the path PA, AQ is less than the time taken by the path PQ. In fact the path PA, AQ is less in time than all other paths like PB, BQ, or PC, CQ, which might be chosen. This leads to the consideration of refractive index and to the law $\sin i / \sin r$ for refraction.

ANALYTICAL GEOMETRY

Fermat wrote a short paper on Analytical Geometry before Descartes produced his *Géométrie*, but it was not published until 14 years after his death. Over some matters the two had disputes. It seems that Fermat discovered how to draw

tangents to curves (letter to Roberval 1636), so as to determine the position of maximum and minimum turning points. His method was equivalent, in modern terminology, to

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

It is stated (Bell) that Newton got some hints of the differential calculus from Fermat's method of drawing tangents. It is even stated by Oystein Ore (*Number Theory and its History*, McGraw Hill, N.Y., 1948) that the French sometimes add the name of Fermat to those of Newton and Leibniz when arguments over priority of the invention of the calculus arise.

PROBABILITY

The story of the problem proposed by the French gambler the Chevalier de Méré, and solved by Pascal and Fermat working together, is variously told. Two players are playing a game with cards, and the first to win two games (or points) out of three is the winner. They are of equal ability and each has put 40 pistoles in the centre for stake. Having won the first game, the Chevalier for some special reason has to stop playing. How, he asks, should the stakes be divided, for neither player could agree on the division which the other proposed?

The problem came to Pascal. He reasoned as follows:

Suppose the second game was played. Then even if the Chevalier lost it, the scores would still be even. Therefore he is entitled to half the 80 pistoles in the centre. Now suppose the third game to be played. De Méré has an even money chance of winning this, so that he is entitled to half of what remains, which is 20 pistoles. Therefore the division should be 60 pistoles to the winner of the first game and 20 to the other, or 3 to 1.

Perhaps because he was not sure of this application of mathematical reasoning to a game of chance, Pascal put the problem to Fermat, who thought it out by permutations.

Look at it from de Méré's point of view, and denote a winning game by W and a losing game by L.

First Game	Second Game	Third Game
W (decided)	W	W
W "	W	L
W "	L	W
W "	L	L

For the first game, which was completed and won by de Méré, he put a W, and then for the next two games he wrote W and L in all the possible ways they could result, without repetition. He obtained the diagram above.

Now since de Méré has only to win one more game, wherever a W appears in the rows the result is favourable to him. Of the four possibilities, three are in his favour and only one, the last, is unfavourable. The odds are therefore 3 to 1 for him, and the stake should be divided 60 pistoles and 20 pistoles, agreeing with Pascal.

Another version (see Von Mises, *Probability Statistics and Truth*, Allen & Unwin, London, 1957, originally in German, Springer, 1928) says that a die was cast four times in succession. One player bet that the 6 would appear at least once, and the other bet against it. The Chevalier de Méré thought there was a slightly better

chance of the 6 coming up at least once in four casts than there was against it. And so it turned out according to Fermat. Von Mises gives the ratio as

$$1 - \left(\frac{5}{6}\right)^4 \text{ to } \left(\frac{5}{6}\right)^4, \text{ or as } 0.5179 \text{ is to } 0.4821.$$

When, however, the game was extended to throwing two dice 24 times, the chance of a double-6 coming up at least once was slightly less than a half, being

$$1 - \left(\frac{35}{36}\right)^{24} \text{ to } \left(\frac{35}{36}\right)^{24}, \text{ or as } 0.4905 \text{ is to } 0.5095.$$

This apparent paradox led to further consideration of the doctrines of Chance, from which eventually arose the systematic study of the theory of probability; another branch of mathematics was born.

The story is often told (see, for example, Newman, *The World of Mathematics*, Simon & Schuster, N.Y., Vol. 1, 1956, p. 375) of G. H. Hardy calling upon Ramanujan lying ill at Putney. By way of conversation, Hardy remarked that the taxi-cab he came in had the number 1729, a dull number he thought, which he hoped was not an unfavourable omen.

"No", replied Ramanujan, "it is a very interesting number, the smallest number expressible as the sum of two cubes in two different ways."

What a splendid conversational opening for a mathematician like Hardy! But he missed the chance and asked only if he, Ramanujan, knew the answer for fourth powers, which is where the topic ended, for Ramanujan did not. Let us imagine the ghost of Pierre Fermat (who could write and speak in English, Spanish, Italian, Latin and Greek, as well as French) calling, nearly three centuries after his death, upon that wonderful Indian mathematician Srinivasa Ramanujan lying seriously ill in London; "Ramanujan, to whom every positive integer was a personal friend" (Littlewood).

We might imagine the following dialogue:

F.: "Allo, Srin! Comment allez-vous ce soir? Well I hope?"

R.: "Monsieur Fermat! How nice of you to come! I think, honoured sahib, I am slowly getting better."

F.: "C'est si bon! So pleased to see you! I came as soon as I heard. I caught a celestial taxi on account of the heavy cloud. The cab, number 1729, had splendid aspects I thought. It might be a good omen!"

R.: "Oh yes, indeed! A very good omen! It is the least possible number expressible as the sum of two cubes in two different ways."

F.: "Ah, you noticed that, then! Did you also observe that it is expressible as the sum of three squares, but not, repeat not, as the sum of two squares?"

R.: "Now that you mention it, it springs to the eye, does it not! And you, vous même, being so interested in primes, must also have noted that it is the product of three primes, namely, 7, 13 and 19."

F.: "I certainly had! One of the first things I thought of! You see, 13 is one of my primes which is of the form $4n+1$. I could not have missed it!"

R.: "Of course not, and therefore it is expressible as the sum of two squares in one way only, 2^2+3^2 , as you have stated, but did not prove."

F.: "A common fault of mine. But Euler proved it later. Il lui donna à penser! You also noted, no doubt, that it is the sum of 14 terms of an arithmetical progression whose first term is 65 and common difference 9?"

R.: "Well no, I did not, but I see that it is true, and my *retort courtoise* is to suggest that it is also the sum of three terms of a geometrical progression whose first term is 130 and whose common ratio is 3. Notice the 3's?"

F.: "Certainement! Knowing you, I am sure you remember the amicable numbers known to the Greeks, 220 and 284. Well I found the second pair in the year 1636, and they were 17,296 and 18,416. Strange is it not that the first number of the pair is my taxi-cab number with the perfect number 6 on the end of it? I saw it at once!"

R.: "Of course I remember the amicable numbers! I also remember that two years later you wrote to Mersenne to tell him you had discovered a third pair. But they were much larger.

F.: "They were, weren't they!"

R.: "I have just thought of something rather odd. You know that there are only four prime numbers between 10 and 20. If one reverses their digits, they remain primes—all except one, 19! When 19 is reversed we get 91, which is not a prime, and ... Ah! you have seen it, of course, 19 times 91 is 1729, my taxi number again!"

F.: "What a pity I did not ask Euler to come. He is with us, you know. I am sure we could have gone on and on. If only we had all lived in the same century what letters we could have written to one another! But I must leave you now, mon ami, my taxi, No. 1729, is still ticking over. Besides, you must get some sleep."

R.: "I am going to talk French, so prenez garde! Vous avez raison, et, merci bien for coming, Monsieur Pierre, you have done me a world of good. Thank you indeed."

F.: "It was my great pleasure, bien entendu. Au revoir, cher ami!"

R.: "Au revoir!"

Had this imaginary conversation taken place, Ramanujan's last words would indeed have been prophetic. He died shortly after that illness at the age of 33, at his home in India.

The concluding lines of Bell's brief but tragic story of the mathematical genius Évariste Galois, who died at the age of 21, shot through the intestines in a sordid unnecessary duel over a woman, are short and heart-rending. They leave the reader with a troubled lump in the throat, and the heart. They are:

"His enduring monument is his collected works. They fill sixty pages."

What might one write to conclude Fermat's story in the same vein? One might write:

His major works comprise a few pencilled notes in the margins of Bachet's translation of Diophantus.