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Babylonian Sexagesimal Reciprocal Tables

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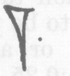
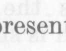
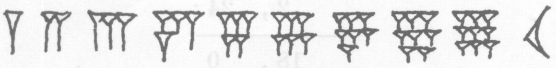
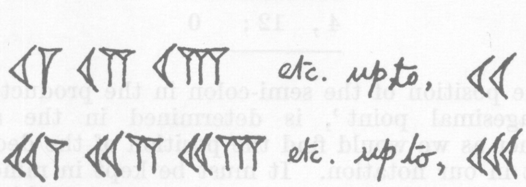
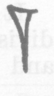
The UTECOM computer at the University of New South Wales was recently used to produce a set of tables of Reciprocals in the Babylonian sexagesimal number system. The table will permit comparison with the 5000-year-old tables that have survived on baked clay tablets, and will be of use in interpreting other tablets which represent calculations performed using tables of Reciprocals as an aid.

The originality of the mathematics of the Babylonians is due primarily to the system of numeration which they inherited from the Sumerians, a non-semitic race of people, who settled in the valley between the Tigris and Euphrates rivers roughly 5000 years ago. The system was sexagesimal, advancing by units of 60, while using only two signs, which they scratched or marked on a wet clay tablet usually about the size of one's hand. They used a stylus sharpened at one end, which, as it was drawn away from the tablet, left a wedge-shaped impression with a long thin "tail" like  This would represent 1, and by pushing the stylus in sideways the indentation looked like  which represented 10. The system was thus in part a decimal one, and the numbers from 1 to 10 were written as follows:

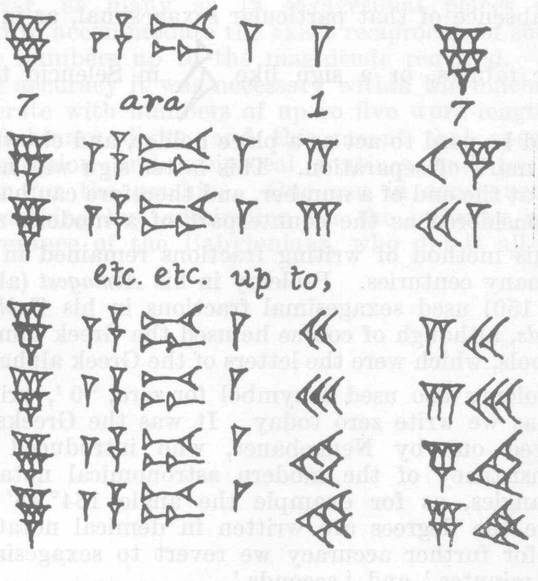


The numbers from 11 to 60 were written

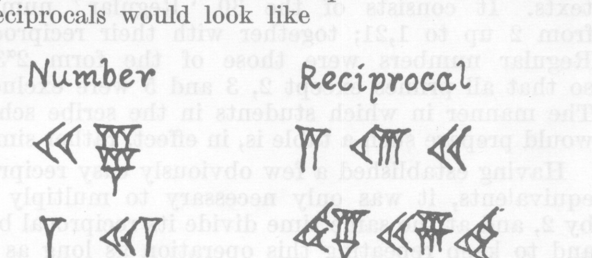


and so on up to 59, which was written  and when 60 was reached they advanced to the next sexagesimal place, so to speak, and wrote there 

Thus the Babylonian multiplication table for 7 times was



A special sign for zero was not used in early Babylonian times. Some examples from their tables of reciprocals would look like



which reads as follows:

Number	Reciprocal
27	;2, 13, 20
1, 21	;0, 44, 26, 40

in our notation, 27 $\quad \frac{2}{60}+\frac{13}{60^2}+\frac{20}{60^3}$

81 $\quad \frac{44}{60^2}+\frac{26}{60^3}+\frac{40}{60^4}$.

The reciprocal tables were probably used as much as the multiplication tables, as well as tables of squares and cubes, much in the same way as a modern mathematician would use a table of logarithms. The reciprocal tables were used when a division operation was to be performed. The scribe would look up the reciprocal of the divisor from his table of reciprocals, and then perform a multiplication with it, using his comprehensive multiplication tables.

In this operation the scribe did not indicate where the 'decimal point' (here represented by a semi-colon) was placed. Neither was it indicated in the product; its position was noted by him in much the same way as is usual in our modern multiplication methods with decimal fractions.

There was similarly no comma between the numbers of the sexagesimals, as we have shown, but merely a space. When a number like 12, 0, ; 35 needed to be written, either a larger space would be left to indicate the absence of that particular sexagesimal, as in the older tablets, or a sign like  in Seleucid texts would be used to act as a place holder, and elsewhere as a mark of separation. This latter sign was never used at the end of a number, and therefore can hardly be considered as the counterpart of a modern zero.

This method of writing fractions remained in use for many centuries. Ptolemy in his *Almagest* (about A.D. 150) used sexagesimal fractions in his *Table of Chords*, although of course he used the Greek number symbols, which were the letters of the Greek alphabet.

Ptolemy also used a symbol for zero, '0', written just as we write zero today. It was the Greeks, as pointed out by Neugebauer, who introduced the inconsistency of the modern astronomical notation for angles, as for example the angle 134° 48′ 27″, where the degrees are written in demical notation, and for further accuracy we revert to sexagesimals for 'minutes' and 'seconds'.

What has been called the Babylonian Standard Table of Reciprocals occurs quite frequently in table texts. It consists of the 30 'Regular' numbers from 2 up to 1,21; together with their reciprocals. Regular numbers were those of the form $2^{\alpha}3^{\beta}5^{\gamma}$, so that all primes except 2, 3 and 5 were excluded. The manner in which students in the scribe schools would prepare such a table is, in effect, rather simple.

Having established a few obviously easy reciprocal equivalents, it was only necessary to multiply one by 2, and at the same time divide its reciprocal by 2, and to keep repeating this operation as long as one chose, and then to do the same with the numbers 3 and 5, or any other number, thus:

Number	Reciprocal
2;	;30
4;	;15
8;	; 7, 30
16;	; 3, 45
32;	; 1, 52, 30
1, 4;	; 0, 56, 45
2, 8;	; 0, 28, 22, 30

Reciprocals of numbers which were not 'Regular', such as 7, 11, 13, etc., are found in some texts. For calculations requiring division by 7, for example, the scribe sometimes used an approximation, as ;8, 34, 16, 59, labelling it 'in defect', or ;8, 34, 18, labelling it 'in excess'.

The reciprocal of 7 in sexagesimals is

;8, 34, 17, 8, 34, 17, 8, 34, 17 . . .

the numbers repeating themselves indefinitely, just as they do in the decimal system, where $\frac{1}{7}$ is

$\cdot\dot{1}42857142857\dot{}$.

The cyclic repetition of the digits 1, 4, 2, 8, 5, 7, which occurs in the decimal system for sevenths, is preserved in the sexagesimal system, and this repetition occurs also for all the other repeating decimals.

Fraction	Decimal	Sexagesimal
$\frac{1}{7}$	·142857	; 8, 34, 17
$\frac{2}{7}$	·285714	;17, 8, 34
$\frac{4}{7}$	·571428	;34, 17, 8
$\frac{3}{7}$	·428571	;25, 42, 51
$\frac{5}{7}$	·714285	;42, 51, 25
$\frac{6}{7}$	·857142	;51, 25, 42

When the Babylonian scribe was required to perform a division, he would find the reciprocal of the divisor from his reciprocal tables and then multiply by it, using his multiplication tables. Thus, in the tablet BM 85200, a division to be performed is $1\cdot75 \div \cdot0069\dot{4}$ in decimal notation, or as the scribe writes it in sexagesimals, 1;45 ÷;0,25. From his tables, the reciprocal of 25 is 2,24, so that the division sum becomes the multiplication 1;45 ×2,24; the answer to which is shown as 4,12; at once, without any of the intermediate steps in the multiplication being shown on the tablet. If the setting out of this multiplication had been given in detail, it would probably have looked as under.

	1 ;	45
	2 ,	24 ;
	18 ,	0
	24	
1 ,	30	
2		
4 ,	12 ;	0

The position of the semi-colon in the product, the 'sexagesimal point', is determined in the same manner as we would find the position of the decimal point in our notation. It must be kept in mind, as we pointed out earlier, that there was nothing to indicate the commas or semi-colons which we use here, and the array of numbers shown above would be exactly the same for the multiplications 1,45; ×2,24; or 1;45 ×2;24 or ;1,45 ×;2,24 or ;1,45 ×2,24; or any other arrangement.

It is because of this method of performing all divisions that the Babylonians made such frequent and efficient use of reciprocal tables. The modern student of Babylonian mathematics thus finds it necessary on many occasions to refer to prepared tables of reciprocals whenever a tablet has obliterated portions or numbers which are not certain, or on which there are (infrequent) scribal errors.

We have therefore prepared Babylonian reciprocal tables of all regular numbers from 1 up to 1,062,882,000. Such tables have not previously been published,* and only one such is at present known to us, that of Prof. O. Neugebauer of Brown University, Rhode Is., U.S.A., whose tables are of regular numbers up to 59, 39, 8, 21, 53, 16, 48. These may appear to be very comprehensive, but so extensive was the use of reciprocals in the Babylonian era that for the adequate treatment of the astronomical tablets, as distinct from the purely mathematical ones, only complete tables would be of real value, at least up to 60^4, and such magnitude seems to be of prohibitive size and cost of printing.

The present tables were prepared entirely by machine, and tabulated in such a form as to permit photographic copying as necessary. On a modern computer, the calculations involved are, of course, fairly routine ones. For example, the conversion of an ordinary decimal fraction to sexagesimal form is effected by multiplying it alternately by 6 and by 10, each time separating off and recording the digit to the left of the decimal point. In actual fact, the reciprocal calculations were performed on the computer in the binary scale and the results were converted to sexagesimals directly.

A preliminary program was used to prepare a table of ' regular ' numbers. This was done by generating all possible multiples of 2, 3 and 5, up to the chosen limit, and then ' sorting ' them in order of magnitude. In the final result these were tabulated both in decimal and sexagesimal form, along with their reciprocals in sexagesimals.

The reciprocals of ' regular ' numbers all, of course, terminate when expressed in sexagesimal form. However, as many as 18 sexagesimal places are needed to accommodate the exact reciprocals of some of the numbers up to the magnitude required. To ensure accuracy it was necessary within the machine to operate with numbers of up to five word-lengths, or 160 binary digits, and this meant that special multiplication and reciprocal routines had to be written for the machine. One can in consequence have nothing but admiration for the industry and perseverance of the Babylonians, who did it all on clay.

* See *Quellen und Studien*, Abteilung A, Band 3, Springer, Berlin, 1935. O. Neugebauer, *Mathematische Keilschrift Texte*.

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