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BABYLONIAN MATHEMATICS*

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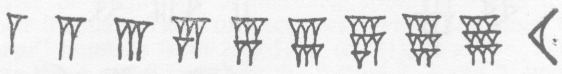
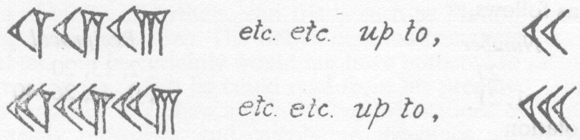
The originality of the mathematics of the Babylonians is due primarily to the system of numeration which they inherited from the Sumerians, a non-semitic race of people, who settled in the valley between the Tigris and Euphrates rivers roughly 5,000 years ago. The system was sexagesimal, advancing by units of 60, while using only two signs, which they scratched or marked on a wet clay tablet usually about the size of one's hand. They used a stylus sharpened at one end, which, as it was drawn away from the tablet, left a wedge shaped impression with a long thin "tail" like  This would represent 1, and by pushing the stylus in sideways the indentation looked like  which represented 10. The system was thus in part a decimal one, and the numbers from 1 to 10 were written as follows,



The numbers from 11 to 60 were written,



and so on up to 59 which was written  and when 60 was reached they advanced to the next sexagesimal place, so to speak, and wrote there  . Thus the Babylonian multiplication table for 7 times was,

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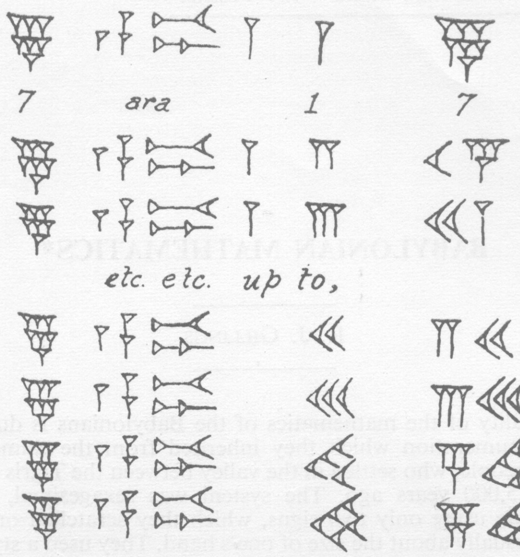
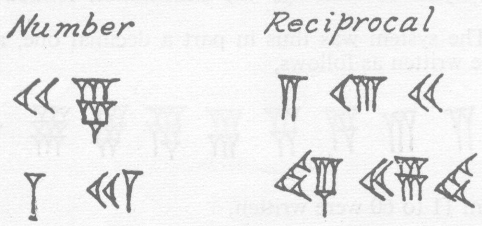
* Address to the Mathematical Association, N.S.W. Branch, 18th June 1954.



A special sign for zero was not used in the early Babylonian period. To examine how Babylonian fractions were written, some examples from their tables of reciprocals will suffice.



which reads as follows,

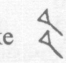
Number	*Reciprocal*
27	; 2, 13, 20
1, 21	; 0, 44, 26, 40

and in our notation

Number	*Reciprocal*
27	$\frac{2}{60} + \frac{13}{60^2} + \frac{20}{60^3}$
81	$\frac{44}{60^2} + \frac{26}{60^3} + \frac{40}{60^4}$.

These reciprocal tables must have been as much used as their multiplication tables, and tables of squares and cubes and others, much in the same way as a modern mathematician would use tables of sines, logarithms, powers, roots, and so on. Of upwards of half a million tablets excavated from Iraq and now mostly reinterred in European, Asian and U.S. museums, only a little more than 300 mathematical texts have been transcribed, translated and published. Of these, about 200 are table texts and about 100 problem texts. The reciprocal tables were always used when a division operation was required, the scribe looking up the reciprocal of the divisor from his table, and then performing a multiplication in which he would use his multiplication tables which were very comprehensive.

In this operation, the scribe did not bother to indicate where the "decimal point" (here indicated by a semi-colon), was placed, he determined its position in his answer, in much the same way as is usual in our modern multiplication of decimal fractions.

When a number such as 2, 0; 35 had to be written, either a space would be left to indicate the absence of a number, as in the older tablets, or a sign written like  in Seleucid texts would be used as a place holder, and elsewhere as a mark of separation, and as it was never used at the end of a number, it can hardly be considered as the counterpart of a modern zero.

This method of writing sexagesimals remained in use for many centuries. It is essentially the same system used by Ptolemy in his "Almagest" (about 150 A.D.), for fractions in his "Table of Chords", although of course he used the Greek number symbols which were the letters of the Greek alphabet with the addition of a symbol for zero, "0", used and written exactly as we use our zero today. It was the Greeks who introduced the inconsistency of the modern astronomical notation for angles, for example 156° 42′ 28″, partly decimal and partly sexagesimal, and where for further accuracy, we revert again to the decimal system for fractions of a second of angle. Of this strangely anomalous system, Neugebauer says, "It took 2,000 years of migration of astronomical knowledge from Mesopotamia via the Greeks, Hindoos and Arabs to arrive at a truly absurd numerical system" (T.E.S.I.A., p. 17).*

Babylonian table texts are extant which give the multiplication tables from 2 to 59, tables of reciprocals of alarming appearance and magnitude, and especially their Standard Table of Reciprocals which contained the reciprocals of 30 "Regular" numbers from 2 to 1, 21 (or 81), as above. These "Regular" numbers were those which had only the factors 2, 3 and 5, numbers in other words of the form $2^{\alpha}3^{\beta}5^{\gamma}$, excluding thus all primes except 2, 3 and 5. Other table texts give not only squares and cubes, but the sum of squares and cubes, and tables of Pythagorean triads such as Plimpton 322, also tables of their weights and measures. The Babylonian mathematician made very good use of his tables, and he certainly would not have bothered to memorise difficult number combinations, when he could read from his prepared tables with ease. The modern tendency to eschew any laborious calculations by using logarithms, slide rules, ready reckoners, and calculating machines would have received the full approbation of the Babylonian scribe. He would have regarded it as a sensible and proper practice.

TABLE TEXTS

Table texts mostly date from the "Old Babylonian" period, about −1900 to −1600. Many of these are the work of apprentice scribes, sometimes showing

* O. Neugebauer, *The Exact Sciences in Antiquity*. Ejnar Munksgaard. Copenhagen 1951.

the master's corrections, or his original table which the student would copy. The manner of preparing the all important "Standard Table of Reciprocals", or in fact any reciprocals, was in effect rather simple. Having established one or two obviously easy reciprocal equivalents, the scribe merely had to multiply one by 2, and divide the other by 2, and keep repeating this operation as long as he chose, referring if need be to his 2 times table or 30 times table. Or he could do the same with 3 or 5.

	Number	*Reciprocal*	
Thus,	2;	; 30	
	4;	; 15	
	8;	; 7, 30	
	16;	; 3, 45	
	32;	; 1, 52, 30	
	1, 4;	; 0, 56, 45	and so on.

Because of his number system, many of the multiplications and divisions would be obvious from observation, for example to halve a number like , he had merely to write down half the number of symbols  and doubling would often be just as simple. The Egyptian scribe was in the same lucky position with his hieroglyphic or hieratic papyri, which of course the Romans and Greeks were not, with their number systems.

Reciprocals of numbers which were not "Regular" such as 7, 11, 13, etc., are found in certain texts and they exhibit very interesting properties. The reciprocal of 7 is found as ; 8, 34, 17, 8, 34, 17, the numbers repeating, just as 142857 does in the decimal system. And the property that $\frac{2}{7}$, $\frac{3}{7}$, . . . are obtainable from $\frac{1}{7}$ by cyclical writing of the digits 142857, is preserved in sexagesimals as follows.

$$\frac{1}{7} = 8,\ 34,\ 17$$
$$\frac{2}{7} = 17,\ 8,\ 34$$
$$\frac{4}{7} = 34,\ 17,\ 8$$
$$\frac{3}{7} = 25,\ 42,\ 51$$
$$\frac{5}{7} = 42,\ 51,\ 25$$
$$\frac{6}{7} = 51,\ 25,\ 42$$

One notices that the numbers of $\frac{3}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$ are the sums of the numbers of $\frac{1}{7}$, $\frac{2}{7}$ and $\frac{4}{7}$, but they still retain their cyclic order. For calculations requiring division by 7 the scribe used the approximation ; 8, 34, 16, 59 labelling it "in deficit" and sometimes ; 8, 34, 18 labelling it "in excess". Table texts may be placed in groups as follows,

- Standard reciprocal tables.
- Tables of reciprocals of regular numbers.
- Tables of reciprocals of irregular numbers.
- Single multiplication tables.
- Combined multiplication tables.
- Tables of square roots.
- Tables of cube roots.
- Tables of logarithms.

The text M.L.C. 2078 (Morgan Library Collection, deposited at Yale University, New Haven, Conn.), is an old Babylonian tablet which answers the question:

To what power must a certain number be raised in order to yield a given number?

The numbers may be written in modern notation as,

$$
\begin{array}{llll}
16 & \text{to the power} & 0;\,15 = & 2 \\
16 & ,, \quad ,, & 0;\,30 = & 4 \\
16 & ,, \quad ,, & 0;\,45 = & 8 \\
16 & ,, \quad ,, & 1;\,0 = & 16 \\
16 & ,, \quad ,, & 1;\,15 = & 32 \\
16 & ,, \quad ,, & 1;\,30 = & 1,\,4
\end{array}
$$

or

$$
\begin{aligned}
\log_{16} 2 &= \tfrac{1}{4} \\
\log_{16} 4 &= \tfrac{1}{2} \\
\log_{16} 8 &= \tfrac{3}{4} \\
\log_{16} 16 &= 1 \\
\log_{16} 32 &= 1\tfrac{1}{4} \\
\log_{16} 64 &= 1\tfrac{1}{2}
\end{aligned}
$$

There is a further set of such tables which uses the equivalent of a base of 2. From these tablets and for other reasons, Neugebauer concludes, "that the old Babylonian mathematicians were very close to an important discovery but failed to take the final essential step". He goes on to suggest that the idea probably originated in the computation of interest, a topic which is dealt with in many previously published texts.

In the tablet Plimpton 322, the oldest extant text on number theory, dating from a period about 1,000 years before Pythagoras, in the Plimpton collection of Columbia University, New York, there is recorded a table of Pythagorean triads. These triads are as under,

Line	1	*b*	*d*	
1	120	119	169	
2	3,456	3,367	4,825	
3	4,800	4,601	6,649	
4	13,500	12,709	18,541	
5	72	65	97	
6	360	319	481	
7	2,700	2,291	3,721	
8	960	799	1,249	(p.)
9	600	481	769	(p.)
10	6,480	4,961	8,161	(p.)
11	60	45	75	
12	2,400	1,679	2,929	
13	240	161	289	
14	2,700	1,771	3,229	(p.)
15	90	56	106	

All except lines 11 and 15 are primitive triads, that is, they are not reducible by division of a common factor. Number 11 is the familiar triad 3, 4, 5 and number 15 reduces to 45, 28, 53. This famous tablet has been the subject of much discussion in recent years. How was such a table constructed? What was the mathematical background of the Babylonian scribes who could find such a primitive triad as 13,500, 12,709, 18,541? Was his method geometrical or algebraic? It has recently been established with reasonable credibility that the scribe used the formula, or its equivalent, which was established 1,300 years later by Euclid in his Elements (BK. 10 Prob. 28 lemma 1),*

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* R. J. Gillings. Unexplained Error in Babylonian Cuneiform Tablet, Plimpton 322. *Aust. Journal of Science.* October 1953.

$$l^2 + b^2 = d^2,$$

where

$$l = 2pq$$
$$b = p^2 - q^2$$
$$d = p^2 + q^2,$$

and where p and q are relatively prime, not simultaneously odd, and p is greater than q.

The values of p and q chosen for the table are all regular numbers, and for the first four triads their values are,

p	q
12	5
64	27
75	32
125	54

One is drawn to the conclusion that Babylonian mathematics was far in advance of anything the contemporary Egyptians developed, the popular stories about the "harpedonaptai" or rope stretchers notwithstanding, these stories being without sound foundations. Archibald (*Outline of the History of Mathematics*, 1949, p. 16), says, "There is no document to prove that the Egyptian knew even a particular case of the Pythagorean Theorem."

The final acceptance of this underlying theory by Assyriologists was primarily due to the existence of certain errors of the text, the interpretation of which is shown to be in conformity with the knowledge of this relation, now a commonplace theorem in modern number theory.

PROBLEM TEXTS

A text BM 13901 of unknown provenance, but dating from before Hammurapi (about -1800), has 20 legible problems and 4 illegible. A sample of these problems is as follows,

$$x^2 + x = ;45$$

$$11x^2 + 7x = 6;15$$

$$\left.\begin{aligned} x^2 + y^2 &= 21,40; \\ x + y &= 50; \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + y^2 &= 21,40; \\ xy &= 10,0; \end{aligned}\right\}$$

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= 23,20; \\ x - y &= 10 \end{aligned}\right\}$$

For each of these problems the scribe directs how the solution is arrived at in the special case under consideration. He does not give any reasons or proofs, nor does he use equations as I am going to do, to indicate his method of solution, but these are the steps which he takes.

I choose a problem from VAT 8389 which reads as follows: "The sum of two quantities is 30, 0; and ; 40 times the first minus ; 30 times the second is 8, 20; What are these numbers?" He has to solve the simultaneous equations,

$$x + y = 30,0; \qquad \text{..........(1)}$$

$$;40x - ;30y = 8,20; \qquad \text{..........(2)}$$

and he directs that the following operations be performed,

Halve 30, 0; = 15, 0;
Take ; 40 − ; 30 = ; 10
Multiply 15, 0; by ; 10 = 2, 30;
Subtract 2, 30; from 8, 20; = 5, 50;
Add ; 40 and ; 30 = 1; 10
Divide 5, 50; by 1; 10 = 5, 0;

Then $x = 15, 0; + 5, 0; = 20, 0;$
and $y = 15, 0; - 5, 0; = 10, 0;$

To illustrate how this method compares with a modern solution, we can change the form of it as follows:

Solve
$$x + y = 8 \quad \text{..........(1)}$$
$$3x + 2y = 21 \quad \text{..........(2)}$$

we have,
$$\tfrac{1}{2}x + \tfrac{1}{2}y = 4 \quad \text{..........(1)}$$
and let,
$$\tfrac{1}{2}x - \tfrac{1}{2}y = K \quad \text{..........(3)}$$

adding,
$$x = 4 + K \quad \text{..........(4)}$$
subtracting,
$$y = 4 - K \quad \text{..........(5)}$$

substitute for x and y in equation (2)
$$3(4 + K) + 2(4 - K) = 21$$
$$12 + 3K + 8 - 2K = 21$$
$$\therefore K = 1.$$

Then from (4) and (5)
$$x = 4 + 1 = 5$$
$$y = 4 - 1 = 3.$$

From tablet BM 13901, the solution of the first problem $x^2 + x = ;45$ amounts to $x = \sqrt{;45 + (\frac{1}{2})^2} - \frac{1}{2}$, and for No. 4 which is of the form $ax^2 + x = c$, the solution is given by
$$x = \frac{\sqrt{ac + (\frac{1}{2})^2} - \frac{1}{2}}{a}$$
and a glance at this shows that the Babylonian mathematician knew the equivalent of the modern form, so well known,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
the solution of the general quadratic equation $ax^2 + bx + c = 0$.